

On Rice's formula for stationary multivariate piecewise smooth processes

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Abstract

Let $X = \{X_t : t \geq 0\}$ be a stationary piecewise continuous \mathbb{R}^d -valued process that moves between jumps along the integral curves of a given continuous vector field, $S \subset \mathbb{R}^d$ be a smooth surface. The aim of this paper is to derive a multivariate version of Rice's formula, relating the intensity of the point process of (localized) continuous crossings of S by X to the distribution of X_0 . Our result is illustrated by examples relating to queueing networks and stress release network models.

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1 Introduction

The classical Rice's formula going back to [10] gives the intensity $\nu(u)$ of crossings (originally, upcrossings) of a given level u by a univariate continuous stationary Gaussian process X_t in terms of the joint distribution of $(X_t, X'_t) \stackrel{d}{=} (X_0, X'_0)$, the process' value and its derivative at a fixed time (provided that the derivative exists in some suitable sense, e.g. in mean quadratic):

$$\nu(u) = \int |z| p(u, z) dz, \quad (1.1)$$

where $p(\cdot, \cdot)$ is the joint density (X_0, X'_0) which is assumed to exist. Later on the result has been extended to more general classes of differentiable (in some suitable sense) stationary processes, covering not only the first moments but also higher order factorial moments of the numbers of crossing, and even to more general settings for continuous random processes and fields. The formula proved to be quite useful in a number of applied areas, including signal processing, reliability, sea waves and others. For detailed accounts of the history of results of this kind and further bibliography, the interested reader is referred to [11], [9] and Chapter 3 in [2].

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The case of processes with jumps and degenerate finite-dimensional distributions drew much less attention, although, from the applications' viewpoint, it is scarcely less interesting than the one of continuous processes. However, the heuristics behind the formula based on “Kac’s counting formula” giving the number of crossings of a level u by a C^1 -function f on $[0, 1]$ as

$$\lim_{\delta \rightarrow 0+} \frac{1}{2\delta} \int_0^1 |f'(t)| \mathbf{1}\{|f(t) - u| < \delta\} dt \quad (1.2)$$

(under a couple of further technical assumptions and denoting by $\mathbf{1}A$ the indicator function of the set A), seems to be applicable in that case as well, provided that the process jumps at finite intensity and is smooth between the jump times. We note that (1.2) is a consequence of Federer’s coarea theorem (see e.g. (7.4.15) in [1])

$$\int_0^1 g(t) |f'(t)| dt = \int_{-\infty}^{\infty} \sum_{s \in [0,1]} g(s) \mathbf{1}\{f(s) = t\} dt, \quad (1.3)$$

applied to the function $g(s) := \mathbf{1}\{|f(s) - u| < \delta\}$.

An analogue of (1.1) for the intensity $\nu_c(u)$ of continuous level crossings by general univariate piecewise deterministic Markov processes that has the form

$$\nu_c(u) = |\mu(u)|p(u),$$

where $\mu(\cdot)$ is the drift coefficient of the process and $p(\cdot)$ the density of X_0 , was established in [6] (see also Theorem 2.5 below; one should mention here an earlier paper [4] where the case of Poisson shot-noise processes was considered). The result was used in [6] to obtain the asymptotic behaviour of the point processes of high level crossings (i.e. as $u \rightarrow \infty$) in a number of interesting and important for applications special cases.

The proof in [6] relied on the Markov structure of the process and in fact did not assume the existence of the density p —its existence was part of the assertion of the main theorem there. The natural question on whether Rice’s formula for piecewise smooth processes can be extended to the multivariate and non-Markovian cases remained open. In the present paper we give a positive answer to it.

The paper is organized as follows. In Section 2, we describe the main class of processes we will be working with and then present the main result together with its proof. Section 3 presents examples to illustrate our main result.

2 The main result

First we will describe the main model of multivariate random processes $X = \{X_t : t \geq 0\}$ dealt with in this paper. The two key elements of the model are a point process $N = \{N(B) : B \in \mathcal{B}(\mathbb{R}_+)\}$ of jumps in our process X (here and in what follows, $\mathcal{B}(\cdot)$ denotes the class of Borel subsets of \cdot) and a vector field $\mu : D \rightarrow \mathbb{R}^d$ defined on an open domain $D \subset \mathbb{R}^d$ and specifying the dynamics of X between the jumps. Note that we allow trivial jumps (of zero size) as well.

We assume that the following assumptions are satisfied.

(A.1) N is a stationary simple counting process on \mathbb{R}_+ , which has a finite intensity $\lambda_N := \mathbb{E}N((0, 1])$ and is such that $N(\mathbb{R}_+) = \infty$ a.s.

The latter implies that the process N is locally finite and hence its points can be enumerated in the increasing order. We denote them by $0 < T_1 < T_2 < \dots$ and set $T_0 := 0$ for convenience (this is not a point of N a.s.).

(A.2) $\mu \in C^1(D)$.

This assumption implies that there exist continuous functions $t_- : \mathbb{R}^d \rightarrow (-\infty, 0)$ and $t_+ : \mathbb{R}^d \rightarrow (0, \infty)$ such that, for any $x \in D$, there exists a unique C^1 -function $q(x, \cdot) : (t_-(x), t_+(x)) \rightarrow D$ satisfying the integral equation

$$q(x, t) = x + \int_0^t \mu(q(x, s)) ds, \quad t \in (t_-(x), t_+(x)) \quad (2.1)$$

(Picard-Lindelöf theorem, see e.g. p.8 in [8]). Moreover, for any fixed $x \in \mathbb{R}^d$ there is a neighbourhood of $(x, 0) \in \mathbb{R}^{d+1}$ in which $q(\cdot, \cdot)$ will also be continuously differentiable (Peano's theorem on dependence on initial conditions, see e.g. p.95 in [8]).

The integral curves q specify the dynamics of the process X between its jumps.

(A.3) Assume that, for any $n \geq 0$, one has $X_{T_n} \in D$, $T_{n+1} < T_n + t_+(X_{T_n})$ and

$$X_t = q(X_{T_n}, t - T_n), \quad T_n \leq t < T_{n+1}. \quad (2.2)$$

Moreover, X and N are jointly stationary, i.e. the distribution of the bivariate process $\{(X_{s+t}, N((s, s+t])) : t \geq 0\}$ does not depend on $s \geq 0$.

Next we will list assumptions involving the surface S of which the continuous crossings by X we are concerned with. The latter are defined as follows: we say that X has a *continuous crossing* of S at time $s > 0$ if $X_{s-} = X_s \in S$ and there is a $\delta > 0$ such that $X_t \notin S$ for $t \in (s - \delta, s + \delta) \setminus \{s\}$.

(A.4) Let $S \subset D$ be the relative interior of a $(d - 1)$ -dimensional (not necessarily connected) C^1 -manifold with or without boundary, and $\{n(x) : x \in S\}$ be a continuous field of unit normals to S . Denoting by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product in \mathbb{R}^d , we assume that

$$\langle n(x), \mu(x) \rangle \neq 0, \quad x \in S. \quad (2.3)$$

Remark 2.1. Let $\tau_x := \inf\{t > 0 : q(x, t) \in S\}$ be the first positive time the integral curve of μ leaving from x at time zero hits the surface S . It is not hard to see that, if S' is a compact subset of S , then from (A.2) and (A.4) it follows that

$$\inf\{\tau_x : x \in S'\} > 0. \quad (2.4)$$

This (together with the fact that X jumps only finitely often in finite time intervals) implies that the times of continuous crossings of S through a compact subset of S cannot accumulate in finite time.

The times of continuous crossings of S by X form an at most countable set N_c that will be identified with a random counting measure on $[0, \infty)$. Then

$$\Phi_c(C) := \sum_{s \in N_c} \mathbf{1}\{(s, X_s) \in C\} = \int \mathbf{1}\{(s, X_s) \in C\} N_c(ds), \quad C \in \mathcal{B}([0, \infty) \times \mathbb{R}^d), \quad (2.5)$$

defines a random (integer-valued) measure Φ_c on $[0, \infty) \times \mathbb{R}^d$. For $t \geq 0$ and $S' \in \mathcal{B}(\mathbb{R}^d)$, the random variable $\Phi_c([0, t] \times S')$ need not be finite. However, if S' is a compact subset of S then (2.4) implies that $\Phi_c([0, t] \times S') < \infty$. Moreover, since N has a finite intensity,

$$\nu_c(B) := \mathbb{E}\Phi_c((0, 1] \times B), \quad B \in \mathcal{B}(\mathbb{R}^d), \quad (2.6)$$

is finite, whenever B is a compact subset of S . Therefore $\nu_c(\cdot)$ is a σ -finite measure on $\mathcal{B}(\mathbb{R}^d)$. For any compact $B \subset S$, the point process $\Phi_c(\cdot \times B)$ is stationary. This is enough to derive the (refined) Campbell theorem stating that

$$\mathbb{E} \int g(s, X_s) N_c(ds) = \iint g(s, x) ds \nu_c(dx). \quad (2.7)$$

for any measurable function $g : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}_+$, cf. e.g. (1.2.19) in [3].

Remark 2.2. Assuming that A is a small enough open set to ensure that $\nu_c(S \cap A) < \infty$, observe that $\nu_c(S \cap A)^{-1} \nu_c(\cdot)$ can be interpreted as the distribution of the value of X at a *typical* time of continuous crossing of $S \cap A$. This is a particular instance of a Palm distribution, see e.g. [3].

The Palm measure π_0 of the pairs of values of X just before and after a typical jump of X is defined by

$$\pi_0(B) := \mathbb{E} \sum_{n=1}^{\infty} \mathbf{1}\{T_n \leq 1, X_{T_n-} \neq X_{T_n}, (X_{T_n-}, X_{T_n}) \in B\}, \quad B \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d). \quad (2.8)$$

Note that $\pi_0(\mathbb{R}^d \times \mathbb{R}^d) \leq \lambda_N < \infty$.

(A.5) The distribution π of X_0 has a continuous density p in a neighborhood of S , and

$$\min\{\pi_0((\mathbb{R}^d \setminus S) \times S), \pi_0(S \times (\mathbb{R}^d \setminus S))\} = 0. \quad (2.9)$$

Now we are ready to state our main result.

Theorem 2.3. *Under assumptions (A.1)–(A.5), one has the identity*

$$\nu_c(B) = \int_{S \cap B} |\langle n(x), \mu(x) \rangle| p(x) \mathcal{H}^{d-1}(dx), \quad B \in \mathcal{B}(\mathbb{R}^d), \quad (2.10)$$

where \mathcal{H}^{d-1} is the $(d-1)$ -dimensional Hausdorff measure on \mathbb{R}^d .

Remark 2.4. As it will be seen from the first half of the proof of Theorem 2.3, the full continuity assumption on p (which is part of (A.5)) can actually be somewhat weakened to the boundedness of p in a neighborhood of S and its right-continuity (in case the first number in (2.9) vanishes) on S along the flow meaning that $p(q(x, 0+)) = p(x)$, $x \in S$.

In the one-dimensional case the above theorem simplifies to the following assertion.

Theorem 2.5. *In the case $d = 1$, assuming that $S = \{u\}$ for some $u \in D$ such that $\mu(u) \neq 0$, and that (A.1)–(A.3) and (A.5) are satisfied, one has*

$$\nu_c(\{u\}) = |\mu(u)|p(u). \quad (2.11)$$

Remark 2.6. In the Markovian case, representation (2.11) was established in [6]. More precisely, it was shown there that there exists a density p satisfying (2.11). Due to the Markovian structure of the process, it was possible to derive the result under weaker technical assumptions.

In the case when

$$S = S^u := \{x \in \mathbb{R}^d : x_1 = u\}$$

for some $u \in \mathbb{R}$, a continuous crossing of S is a continuous crossing of the level u by the first component of X . In this case Theorem 2.3 takes the following form.

Theorem 2.7. *Let assumptions (A.1)–(A.3) be satisfied and $u \in \mathbb{R}$ be such that $S^u \subset D$ and $\mu_1(x) \neq 0$ for all $x \in S^u$, where μ_1 is the first component of μ . Assume that (A.5) holds with $S = S^u$. Then, for $B \in \mathcal{B}(\mathbb{R}^d)$,*

$$\nu_c(B) = \int \cdots \int \mathbf{1}_B(u, x_2, \dots, x_d) |\mu_1(u, x_2, \dots, x_d)| p(u, x_2, \dots, x_d) dx_2 \cdots dx_d.$$

Remark 2.8. Theorem 2.7 is another, more straightforward, generalization of (2.11). Assume now that $0 < \nu_c(S^u) < \infty$ and consider a “typical time” of a continuous crossing of the level u by the first component of X . Then the measure $\mathbb{Q}_u(\cdot) := \nu_c(S^u)^{-1} \nu_c(\{u\} \times \cdot)$ describes the distribution of the other components of X at this time. This distribution can be interpreted in terms of the drift-modulated density p_1 proportional to $|\mu_1(x)|p(x)$ (assuming that $\mathbb{E}|\mu_1(X_0)| < \infty$). If (Y_1, \dots, Y_d) is a random vector with density p_1 , then \mathbb{Q}_u is the conditional distribution of (Y_2, \dots, Y_d) given that $Y_1 = u$.

Remark 2.9. Let $k \in \{1, \dots, d\}$ and assume that $S = \tilde{S} \times \mathbb{R}^{d-k}$, where $\tilde{S} \subset \mathbb{R}^k$ is a $(k-1)$ -dimensional smooth surface. Let $\{\tilde{n}(x) : x \in \tilde{S}\}$ be a continuous field of unit normals to \tilde{S} . Let $\tilde{X} := (X^{(1)}, \dots, X^{(k)})$ and $Y := (X^{(k+1)}, \dots, X^{(d)})$, where $X = (X^{(1)}, \dots, X^{(d)})$. There is a one-to-one correspondence between the continuous crossings of S by the process X and continuous crossings of \tilde{S} by the process \tilde{X} . Equation (2.10) can be written as

$$\nu_c(B) = \int_{\mathbb{Y}} \int_{\tilde{S}} |\langle \tilde{n}(x), \tilde{\mu}(x, y) \rangle| \mathbf{1}_B(x, y) p(x|y) \mathcal{H}^{k-1}(dx) \mathbb{P}(Y_0 \in dy), \quad B \in \mathcal{B}(\mathbb{R}^k \times \mathbb{Y}), \quad (2.12)$$

where $\mathbb{Y} := \mathbb{R}^{d-k}$, $\tilde{\mu}(x, y)$ is the vector of the first k components of $\mu(x, y)$, and $x \mapsto p(x|y)$ is the conditional density of \tilde{X}_0 given that $Y_0 = y$. In this form the result might be generalizable to other stationary pairs (\tilde{X}, Y) . The process \tilde{X} should remain piecewise deterministic for given Y . But the process Y might take values in a more general space \mathbb{Y} . In this paper we will make no attempt to establish such an extension of our results.

To prove Theorem 2.3, we will need an auxiliary result that requires some further notation. First of all, for our purposes it will suffice that that result would hold in a “local setting”, i.e. for S replaced with $S \cap A$, where A is a small enough open subset of \mathbb{R}^d . As can easily be seen from the observation that we made after stating assumption (A.2) and from (A.4), if we understand by S such a “small piece” of the original surface, then the following will be satisfied:

(A.6) The surface S is connected and relatively compact, (2.4) holds with $S' = S$ and $\nu_c := \nu_c(S) < \infty$. Furthermore, there exists a $u_0 > 0$ such that $t_+(x) \geq u_0$ for all $x \in S$ and, for any $u \in [0, u_0]$,

$$S_u := \{q(x, u) : x \in S\}$$

is a C^1 -surface with a continuous field $\{n_u(x) : x \in S_u\}$ of unit normals to it satisfying

$$\inf \{ \langle n_u(x), \mu(x) \rangle : x \in S_u, u \in [0, u_0] \} > 0. \quad (2.13)$$

Moreover, π has a density p in a neighbourhood of $S_{(0, u_0)}$, where

$$S_I := \bigcup_{u \in I} S_u, \quad I \subset \mathbb{R}.$$

Now denote by N_c^u the stationary point process of the times of all continuous crossings of S_u by X . For any $C \in \mathcal{B}([0, \infty) \times \mathbb{R}^d)$, let $\Phi_c^u(C)$ be the number of all $s \in N_c^u$ such that $(s, X_s) \in C$ and $\nu_c^u(B) := \mathbb{E} \Phi_c^u([0, 1] \times B)$, $B \in \mathcal{B}(\mathbb{R}^d)$.

Proposition 2.10. *Under assumptions (A.1)–(A.4) and (A.6), for any measurable function $g : \mathbb{R}^d \rightarrow \mathbb{R}_+$, one has*

$$\int g(x) \nu_c^u(dx) = \int_{S_u} |\langle n(x), \mu(x) \rangle| g(x) p(x) \mathcal{H}^{d-1}(dx) \quad (2.14)$$

for \mathcal{H}^1 -almost all $u \in [0, u_0]$.

PROOF. For any $j \geq 0$ set $T'_j := T_j \wedge 1$ and, in particular, $T'_0 := 0$. For $j \geq 1$ we define

$$I_j := (T'_{j-1}, T'_j), \quad L_j := \{X_t : t \in I_j\} = \{q(X_{T'_{j-1}}, t - T'_{j-1}) : t \in I_j\}.$$

Fix a $B \in \mathcal{B}(\mathbb{R}^d)$ and assume that $u \in (0, u_0)$. By definition, $\Phi_c^u(I_j \times B) > 0$ if and only if $L_j \cap S_u \cap B \neq \emptyset$. On the other hand, (2.13) implies that $\Phi_c^u(I_j \times B) \leq 1$, so that

$$\Phi_c^u(I_j \times B) = \mathbf{1}\{L_j \cap S_u \cap B \neq \emptyset\}.$$

Therefore

$$\Phi_c^u((0, 1) \times B) = \sum_{j=1}^{\infty} \mathbf{1}\{L_j \cap S_u \cap B \neq \emptyset\}$$

and, for any $v \in (0, u_0)$,

$$\int_0^v \Phi_c^u((0, 1) \times B) du = \sum_{j=1}^{\infty} \int_0^v \mathbf{1}\{L_j \cap S_u \cap B \neq \emptyset\} du. \quad (2.15)$$

Now set

$$J_j(v) := \{t \in I_j : X_t \in S_{(0,v)}\}, \quad U_j(v) := \{u \in (0, v) : L_j \cap S_u \neq \emptyset\}.$$

Clearly, the last two sets are either simultaneously empty or are open intervals of the same length; in the latter case, put $u_j(v) := \inf U_j(v)$. Therefore,

$$\begin{aligned} \int_0^v \mathbf{1}\{L_j \cap S_u \cap B \neq \emptyset\} du &= \int_{U_j(v)} \mathbf{1}\{q(X_{u_j(v)}, u - u_j(v)) \in B\} du \\ &= \int_{J_j(v)} \mathbf{1}\{X_t \in B\} dt = \int_{I_j(v)} \mathbf{1}\{X_t \in S_{(0,v)} \cap B\} dt, \end{aligned} \quad (2.16)$$

so that (2.15) becomes

$$\int_0^v \Phi_c^u((0, 1) \times B) du = \int_0^1 \mathbf{1}\{X_t \in S_{(0,v)} \cap B\} dt.$$

Taking expectations on both sides of the last relation and using Fubini's theorem and stationarity of X , we obtain that

$$\int_0^v \nu_c^u(B) du = \mathbb{E} \int_0^1 \mathbf{1}\{X_t \in S_{(0,v)} \cap B\} dt = \mathbb{P}(X_0 \in S_{(0,v)} \cap B).$$

As functions of $B \in \mathcal{B}(\mathbb{R}^d)$, both sides specify a measure, and so the standard argument shows that, for any measurable function $g : \mathbb{R}^d \rightarrow \mathbb{R}_+$,

$$\int_0^v du \int g(x) \nu_c^u(dx) = \int_{S_{(0,v)}} g(x) p(x) \mathcal{H}^d(dx). \quad (2.17)$$

Now we can assume without loss of generality that S admits a C^1 -parametrization $(w_1, \dots, w_{d-1}) \mapsto z(w_1, \dots, w_{d-1})$, where (w_1, \dots, w_{d-1}) varies in an open set $W \subset \mathbb{R}^{d-1}$. For $(w_1, \dots, w_{d-1}) \in W$ and $u \in [0, u_0]$, define

$$\psi(w_1, \dots, w_{d-1}, u) := q(z(w_1, \dots, w_{d-1}), u),$$

which, for a fixed $u \in [0, u_0]$, will be a C^1 -parametrization of the “parallel” surface S_u .

Next we denote by ∂_i the operator of partial differentiation with respect to w_i , $i = 1, \dots, d-1$, and let $\partial_d \psi := \partial/\partial u$. A simple linear algebra calculations shows that the Jacobian $J\psi$ of $\psi = \psi(w_1, \dots, w_{d-1}, u)$ satisfies

$$|J\psi| = |\langle n_u(\psi), \partial_d \psi \rangle| H \equiv |\langle n_u(\psi), \mu(\psi) \rangle| H,$$

where $H^2 = H^2(w_1, \dots, w_{d-1}, u)$ is the determinant of the matrix $(\langle \partial_i \psi, \partial_j \psi \rangle)_{i,j=1,\dots,d-1}$. However, for any fixed $u \in [0, u_0]$, $H(w_1, \dots, w_{d-1}, u) dw_1 \cdots dw_{d-1}$ is the surface element of S_u in the coordinates (w_1, \dots, w_{d-1}) , so that changing coordinates on the right-hand side of (2.17) yields

$$\begin{aligned} \int_{S_{(0,v)}} g(x) p(x) \mathcal{H}^d(dx) &= \int_{W \times (0,v)} g(\psi) p(\psi) |J\psi| dw_1 \cdots dw_{d-1} du \\ &= \int_{W \times (0,v)} g(\psi) p(\psi) |\langle n_u(\psi), \mu(\psi) \rangle| H dw_1 \cdots dw_{d-1} du \\ &= \int_0^v du \int_{S_u} g(x) p(x) |\langle n_u(x), \mu(x) \rangle| \mathcal{H}^{d-1}(dx), \end{aligned} \quad (2.18)$$

which immediately implies the assertion of Proposition 2.10. \square

Remark 2.11. Assume that f is a real-valued C^1 -function defined on an open domain $\tilde{D} \subset \mathbb{R}^d$, with non-vanishing gradient and such that $S_u = \{x \in \tilde{D} : f(x) = u\}$ for all small enough u . Such a function exists, at least for suitably small pieces of S . We may then apply Federer's coarea theorem (see e.g. (7.4.15) in [1]) on each open interval (T'_{j-1}, T'_j) to the level sets of the function $t \mapsto f(X_t)$. While this would provide an alternative way for deriving (2.16), we have preferred to give a direct argument presented in the above proposition. In a quite similar spirit the coarea theorem can be used to derive Rice's formula for smooth processes, see Section 11.4 in [1]. It was actually U. Zähle who first used in [12] the coarea theorem to prove Rice's formula for certain continuous processes. We also note in passing that the coarea formula could be used to establish (2.18) as well. However, our more explicit argument yields additional information that is needed in the proof of Theorem 2.3.

PROOF OF THEOREM 2.3. Since both sides of (2.10) are σ -additive in B , it is no restriction of generality to assume that assumption **(A.6)** is satisfied. Moreover, we can assume that S admits a smooth parametrization as in the proof of Proposition 2.10. This is due to the fact that the surface S can be represented as a “mosaic” of “small pieces” for which the assumption will be satisfied owing to assumptions **(A.1)–(A.4)** on the original S .

Furthermore, it is not hard to see that, to prove the theorem, it suffices to demonstrate that (2.14) holds at $u = 0$ for continuous and bounded g . We will show that by proving that, under the assumption $\pi_0((\mathbb{R}^d \setminus S) \times S) = 0$, both sides of (2.14) are right-continuous at $u = 0$, as Proposition 2.10 will imply then the desired result. The case when only the second term on the left-hand side of (2.9) turns into zero (i.e. $\pi_0(S \times (\mathbb{R}^d \setminus S)) = 0$) can be dealt with in exactly the same way by establishing the left-continuity of both sides of (2.14) at $u = 0$ in this situation (essentially via a time-reversal argument).

Using the notation from the proof of Proposition 2.10 and setting

$$h_u(w) := \langle n_u(\psi), \mu(\psi) \rangle g(\psi) p(\psi), \quad w = (w_1, \dots, w_{d-1}), \quad \psi = \psi(w, u),$$

we have, for $u \in [0, u_0]$,

$$\int_{S_u} \langle n_u(x), \mu(x) \rangle g(x) p(x) \mathcal{H}^{d-1}(dx) = \int h_u(w) H(w, u) \mathcal{H}^{d-1}(dw). \quad (2.19)$$

As noted after stating assumption **(A.2)**, one has $q \in C^1$, and so $n_u(\psi(y, u))$ is a continuous function of u , leading to

$$h_{0+}(w) = \langle n(z(w)), \mu(z(w)) \rangle g(z(w)) p(z(w)).$$

Similarly, as $u \rightarrow 0+$, $H^2(w, u)$ converges to the value of the determinant of the matrix $(\langle \partial_i z(w), \partial_j z(w) \rangle)_{i,j=1,\dots,d-1}$. Now the dominated convergence theorem implies that (2.19) converges to the right-hand side of (2.14) at $u = 0$.

To establish the desired right-continuity of the left-hand side of (2.14) we assume that (2.13) holds. Introduce the following point process Φ_d on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$:

$$\Phi_d(\cdot) := \sum_{n=1}^{\infty} \mathbf{1}\{X_{T_n-} \neq X_{T_n}\} \mathbf{1}\{(T_n, X_{T_n-}, X_{T_n}) \in \cdot\}.$$

Let $u \in [0, u_0]$ and $t \geq 0$. A continuous crossing of S_u can only occur on a trajectory of X that arrives at the surface from the *inside* of $S_{(0,u)}$ along an integral curve of μ (cf. (2.13)). Therefore each such crossing of S_u should be preceded by an entry to $S_{[0,u]}$, either along a drift line or by jump. Taking into account the possibility of having $X_0 \in S_{[0,u]}$, we obtain the bound

$$N_c^u([u, t+u]) \leq N_c([0, t+u]) + \Phi_d([0, t+u] \times (\mathbb{R}^d \setminus S_{[0,u]}) \times S_{[0,u]}) + 1.$$

Therefore

$$\begin{aligned} \int_u^{t+u} g(X_s) N_c^u(ds) &\leq \int_0^{t+u} (g(X_s) + \varepsilon(u)) N_c(ds) \\ &\quad + g^* [\Phi_d([0, t+u] \times (\mathbb{R}^d \setminus S_{[0,u]}) \times S_{[0,u]}) + 1], \end{aligned}$$

where $g^* := \sup_x g(x)$ and

$$\varepsilon(u) := \sup\{|g(q(x, v)) - g(x)| : x \in S, 0 \leq v \leq u\} \rightarrow 0 \quad \text{as } u \rightarrow 0+ \quad (2.20)$$

due to the uniform continuity of the mapping $(x, u) \mapsto g(q(x, u))$ on $\bar{S} \times [0, u_0]$, \bar{S} denoting the closure of S .

Now taking expectations on both sides of the obtained inequality and using Campbell's formula (2.7) yields

$$t \int g(x) \nu_c^u(dx) \leq (t+u) \int (g(x) + \varepsilon(u)) \nu_c(dx) + g^* ((t+u) \pi_0((\mathbb{R}^d \setminus S_{[0,u]}) \times S_{[0,u]}) + 1),$$

where we also used Campbell's theorem for $\pi_0(\cdot) = \mathbb{E} \Phi_d([0, 1] \times \cdot)$. After dividing by t and letting $t \rightarrow \infty$, we obtain

$$\int g(x) \nu_c^u(dx) \leq \int g(x) \nu_c(dx) + \varepsilon(u) \nu_c(S) + g^* \pi_0((\mathbb{R}^d \setminus S_{[0,u]}) \times S_{[0,u]}).$$

In view of (2.20) and the fact that the assumption $\pi_0((\mathbb{R}^d \setminus S) \times S) = 0$ implies that $\pi_0((\mathbb{R}^d \setminus S_{[0,u]}) \times S_{[0,u]}) \rightarrow 0$ as $u \rightarrow 0+$, this leads to

$$\limsup_{u \rightarrow 0+} \int g(x) \nu_c^u(dx) \leq \int g(x) \nu_c(dx).$$

To derive the converse inequality, we start with the observation that any continuous crossing of S in $[0, t]$ is followed either by a continuous crossing of S_u or by a jump from $S_{(0,u]}$ to its complement within the time interval $[0, t+u]$, so that

$$N_c^u([0, t+u]) \geq N_c([0, t]) - \Phi_d([0, t+u] \times S_{(0,u]} \times (\mathbb{R}^d \setminus S_{(0,u]})) - 1.$$

Next, similarly to our argument above, we obtain

$$\int g(x) \nu_c^u(dx) \geq \int g(x) \nu_c(dx) - \varepsilon(u) \nu_c(S) - g^* \pi_0(S_{(0,u]} \times (\mathbb{R}^d \setminus S_{(0,u]})).$$

Since $\lim_{u \rightarrow 0+} S_{(0,u]} = \emptyset$, it is clear that the continuity of π_0 implies now that

$$\liminf_{u \rightarrow 0+} \int g(x) \nu_c^u(dx) \geq \int g(x) \nu_c(dx),$$

which completes the proof of the theorem. \square

3 Examples

In this section we will present two examples showing possible uses of our main result.

Example 3.1. Consider a general queueing network model with d servers operating in stationary regime, with arrivals of customers (possibly in batches) to the network being governed by a stationary simple point process. Each customer, upon completion of its service at node $j \in \{1, \dots, d\}$ of the network, proceeds to another node for further service or leaves the network, according to some routing mechanism. All the arrival, transition and departure times form a stationary point process N , and it is at these times that the state of the process $X_t = (X_t^{(1)}, \dots, X_t^{(d)}) \in \mathbb{R}^d$ describing the residual workloads on the nodes can change by a jump. Between the events, the values of X_t decrease according to the relation $\frac{d}{dt}X_t = \mu(X_t)$ for some C^1 -function $\mu : \mathbb{R}^d \rightarrow \mathbb{R}_-^d$, so that the service rate at node j can depend on the residual workload at the node and, moreover, it can even depend on the workloads at other nodes $i \neq j$ as well. To make this description compatible with the assumptions in Section 2, we allow $X_t^{(j)} < 0$ interpreting as the residual workload at node j at time t the value $\max\{X_t^{(j)}, 0\}$, and let $D := \mathbb{R}^d$.

For $i \in \{1, \dots, d\}$ let $S_i := \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : x_i = 0\}$. Then the continuous crossing of the surface S_i corresponds to server i becoming idle. Let $\nu_i(B)$ denote the intensity of these crossings through a point in $B \in \mathcal{B}(\mathbb{R}^d)$. Provided that the assumptions of Theorem 2.7 are satisfied, we obtain

$$\nu_i(B) = \int \cdots \int \mathbf{1}_B(x^i) |\mu_i(x^i)| p(x^i) dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_d, \quad (3.1)$$

for $B \in \mathcal{B}(\mathbb{R}^d)$, where $x^i := (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_d)$ and μ_i is the i th component of μ . The normalization of (3.1) yields the (Palm) distribution of the network at a typical departure time from node i .

Note that assumptions (A.1)–(A.3) are rather mild and that Theorem 2.7 also requires $\mu_i(x) < 0$ for $x \in S_i$. In assumption (2.9) only the condition $\pi_0(S_i \times (\mathbb{R}^d \setminus S_i)) = 0$ is of relevance. This assumption says that if there is a jump at an instant when server i becomes empty, then the workload of this server is not allowed to increase by this jump, neither by an internal transition (including feedback) nor by an external arrival. Again, this is a rather weak assumption.

We can also consider the “composite surface” $S := \bigcup_i S_{(i)}$, where $S_{(i)}$ is the set of all $x \in \mathbb{R}^d$ with $x_i = 0$ and $x_j \neq 0$ for $j \neq i$. (Under weak assumptions any continuous crossing of S_i is also a continuous crossing of $S_{(i)}$.) Theorem 2.7 provides the Palm distribution of the residual workloads at the time when one of the servers becomes idle while all the others are still working. The probability of server i becoming idle given a typical instant when (exactly) one of the servers becomes idle is then given by $\nu_c(S_{(i)})/\nu_c(S)$. We skip further details.

Example 3.2. The classical stress release model in seismology (see e.g. [7] and references to earlier work therein) is a piecewise deterministic Markov process X_t representing the level of “stress” at a seismic fault at time t . The value X_t continuously increases at a linear rate due to the tectonic loading of the fault and drops by random jumps when the stress discharges by way of earthquakes that occur at random times whose intensity is given by $\psi(X_t)$ for some suitably chosen increasing risk function ψ (e.g. $\psi(x) = e^{\beta x}$

for some $\beta > 0$). Note that the remote measuring of stress levels at seismic faults is an extremely difficult problem, so the value X_t is usually not observable. All the information on the process one can have access to is contained in the times, locations and magnitudes of jumps.

A more interesting multinode analog of the model was discussed in [5], where it was demonstrated, in particular, that already a two-node stress release network can reproduce the famous Omori's law for the intensity of earthquake aftershocks.

In the multinode model, the values of the components of the random process $X_t = (X_t^{(1)}, \dots, X_t^{(d)}) \in \mathbb{R}^d$ represent the time t stress levels at individual seismic faults $j \in \{1, \dots, d\}$ constituting a local fault system. Between jumps, the dynamics of the process are given by $\frac{d}{dt}X_t = \mu$ for a constant vector $\mu \in \mathbb{R}^d$. Note that one can have $\mu_j < 0$ which corresponds to tectonic unloading of stress at node j (of course, we can consider a more general model with a variable μ as well; similar remarks apply to all the other elements of the model construction). Jumps ("seismic events") occurrence at node j is driven by a Markovian random mechanism with the probability of a jump occurring at the node in the infinitesimal time interval dt given by $\psi_j(X_{t-})dt$ for a given risk function $\psi_j(x)$.

When the n th seismic event occurs at node j (say, at time $T_{j,n}$), the value of stress at the node changes by a random quantity $\xi_{j,n}$, $n = 1, 2, \dots$, which may be assumed to be i.i.d. random variables. Moreover, the stress levels at other nodes can also change at that instance: for a given constant transfer matrix $(r_{ij}) \in \mathbb{R}^{d \times d}$, one has $X_{T_{j,n}}^{(i)} = X_{T_{j,n}-}^{(i)} + r_{ij}\xi_{j,n}$, $i \neq j$ (for more detail, see [5]).

One of the main problems one hopes to be able to solve in mathematical seismology is to give advanced earthquake warnings. Within the framework of the multinode stress release model, that warning would have to be given at the time when the cumulative jump intensity $\sum_{j=1}^d \psi(X_t^{(j)})$ exceeds a given threshold $u > 0$. That is, we are looking at continuous crossings of the surface $S := \{x : \sum_{j=1}^d \psi(x_j) = u\}$ by our process X_t . Our main result allows to find the distribution of X_t at the (typical) time of such crossing and hence, for example, to derive the probability for a given fault to trigger the forthcoming seismic event.

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